

# THREE-DIMENSIONAL ISOLATED QUOTIENT SINGULARITIES IN EVEN CHARACTERISTIC

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**ABSTRACT.** This paper is a complement to the work of the second author on modular quotient singularities in odd characteristic. Here we prove that if  $V$  is a three-dimensional vector space over a field of characteristic 2 and  $G < \mathrm{GL}(V)$  is a finite subgroup generated by pseudoreflections and possessing a 2-dimensional invariant subspace  $W$  such that the restriction of  $G$  to  $W$  is isomorphic to the group  $\mathrm{SL}_2(\mathbb{F}_{2^n})$ , then the quotient  $V/G$  is non-singular. This, together with earlier known results on modular quotient singularities, implies first that a theorem of Kemper and Malle on irreducible groups generated by pseudoreflections generalizes to reducible groups in dimension three, and, second, that the classification of three-dimensional isolated singularities which are quotients of a vector space by a linear finite group reduces to Vincent's classification of non-modular isolated quotient singularities.

## 1. INTRODUCTION

Let  $k$  be a field of characteristic  $p$  and  $V$  a finite dimensional vector space over  $k$ . A linear map  $\varphi: V \rightarrow V$  is called a *pseudoreflection* if the set of points fixed by  $\varphi$  is a hyperplane in  $V$ . A pseudoreflection  $\varphi$  is called a *transvection* if 1 is the only eigenvalue of  $\varphi$ . Denote by  $V^*$  the dual space and by  $S(V^*)$  its symmetric algebra. In [7] Kemper and Malle proved the following theorem.

**Theorem 1.1.** *Let  $G$  be a finite irreducible subgroup of  $\mathrm{GL}(V)$ . Then its ring of invariants  $S(V^*)^G$  is polynomial if and only if  $G$  is generated by pseudoreflections and the pointwise stabilizer in  $G$  of any non-trivial subspace of  $V$  has a polynomial ring of invariants.*

Kemper and Malle also asked if the condition “irreducible” could be eliminated from the statement of their theorem. They showed that to obtain such a generalization it is sufficient to investigate the general reducible but non-decomposable case and pointed out that the generalized theorem holds in dimension 2. Note that the direct statement of Theorem 1.1 (“if the ring  $S(V^*)$  is polynomial, then ...”) is correct without the condition of irreducibility; it follows from the Chevalley-Shephard-Todd Theorem if  $p$  does not divide the order of  $G$ , and in the modular case  $p \mid |G|$  it was proven by Serre.

From the perspective of singularity theory, Stepanov in [8] showed that if the generalized (to reducible groups  $G$ ) theorem of Kemper and Malle is correct, it can be interpreted as saying that each isolated singularity which is a quotient of a vector space by a finite modular linear group is in fact isomorphic to a quotient by a non-modular group. Thus the classification of such singularities reduces to the known Vincet's classification of isolated quotient singularities in the non-modular case; for the details, see [8] and references therein. Stepanov started also studying 3-dimensional case and obtained the following result.

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**Theorem 1.2** ([8, Theorem 4.1]). *Let  $V$  be a 3-dimensional vector space over an algebraically closed field of characteristic  $p$ . Let  $G$  be a finite subgroup of  $GL(V)$  generated by pseudoreflections. Denote by  $G_p$  the normal subgroup of  $G$  generated by all elements of order  $p^r$ ,  $r \geq 1$ . Assume that  $G_p$  is either*

- (1) *irreducible on  $V$ , or*
- (2) *has a 1-dimensional invariant subspace  $U$ , or*
- (3) *has a 2-dimensional invariant subspace  $W$  and the restriction of  $G_p$  to  $W$  is generated by two non-commuting transvections (and thus is irreducible).*

*Then the generalized Kemper-Malle Theorem holds for  $G$ . Moreover, if  $G$  satisfies condition (3) or condition (2) plus the induced action of  $G_p$  on  $V/U$  is generated by two non-commuting transvections, then  $V/G$  is non-singular.*

Note that if a map  $\varphi \in GL(W)$ ,  $\dim W = 2$ , has order  $p^r$ ,  $r \geq 1$ , then it has order  $p$  and is a transvection. In view of the classification of 2-dimensional groups generated by transvections, Theorem 1.2 applies to all modular groups in odd characteristic. In characteristic 2 it remains to consider only the case when  $G$  has a 2-dimensional invariant subspace  $W$  and the restriction  $H$  of  $G_2$  to  $W$  is isomorphic to the group  $SL_2(\mathbb{F}_{2^n})$  (the group of all  $2 \times 2$  matrices of determinant 1 with entries in the Galois field with  $2^n$  elements),  $n > 1$ , in its natural representation.

In the present paper we fill this gap and show, moreover, that no singularities arise in the remaining case  $H = SL_2(\mathbb{F}_{2^n})$ ,  $n > 1$ . Our main result is Theorem 1.3 below. As was shown in [8], we can assume from the beginning that  $G = G_2$  and the base field  $k$  is algebraically closed.

**Theorem 1.3.** *Let  $V$  be a 3-dimensional vector space over an algebraically closed field  $k$  of characteristic 2. Let  $G$  be a finite subgroup of  $GL(V)$  generated by pseudoreflections of order  $2^r$ ,  $r \geq 1$ , and hence by transvections. Assume that  $G$  has a 2-dimensional invariant subspace  $W$  and the restriction of  $G$  to  $W$  is isomorphic to the group  $SL_2(\mathbb{F}_{2^n})$ ,  $n > 1$ , in its natural representation. Then the ring of invariants  $S(V^*)^G$  is polynomial.*

*Remark 1.4.* It follows from our results that if  $G < GL(V)$ ,  $\dim V = 3$ , characteristic is arbitrary, is any finite subgroup generated by pseudoreflections and possessing a 2-dimensional invariant subspace or a 1-dimensional invariant subspace satisfying the additional condition of Theorem 1.2, then the quotient  $V/G$  is non-singular. However, it is not true that Chevalley-Shephard-Todd Theorem holds for modular groups in dimension 3. In [7] Kemper and Malle give examples of *irreducible* groups  $G$  generated by pseudoreflections for which the ring  $S(V^*)^G$  is not polynomial. In dimension 4, there are examples (see [5, Example 11.0.3]) of reducible groups generated by pseudoreflections with singular quotients. For general reducible 3-dimensional groups  $G$  generated by pseudoreflections, we do not know if the quotient  $V/G$  can be singular.

As we explained above, our results and Theorem 1.1 of Kemper and Malle imply the following corollaries.

**Corollary 1.5.** *The generalized Kemper-Malle Theorem holds in dimension 3, i.e., if  $V$  is a 3-dimensional vector space and  $G < GL(V)$  is any finite subgroup, then the ring of invariants  $S(V^*)^G$  is polynomial if and only if  $G$  is generated by pseudoreflections and the pointwise stabilizer in  $G$  of any non-trivial subspace of  $V$  has a polynomial ring of invariants.*

**Corollary 1.6.** *If  $V$  is a 3-dimensional vector space over an arbitrary field  $k$ , and  $G$  a finite subgroup of  $GL(V)$  such that the variety  $V/G$  has isolated singularity, then  $V/G$  is isomorphic to one of the non-modular isolated quotient singularities from Vincent's classification.*

We prove our Theorem 1.3 by a more or less direct computation of the ring of invariants of the group  $G$ . The proof is contained below in Sections 2 and 3.

## 2. PROOF OF THEOREM 1.3: THE GROUP $G$ AS AN EXTENSION OF $\mathrm{SL}_2(\mathbb{F}_{2^n})$

Assume that a group  $G$  satisfies the conditions of Theorem 1.3, i.e.,  $G$  is generated by transvections, acts on a 3-dimensional vector space  $V$  with a 2-dimensional invariant subspace  $W$ , and the restriction of  $G$  to  $W$  is isomorphic to the natural action of the group  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  on the space  $k^2$  of column vectors. We shall fix a basis  $(e_1, e_2, e_3)$  of  $V$  such that  $e_1$  and  $e_2$  span  $W$  and each element of the group  $G$  is represented in this basis by a matrix

$$\begin{pmatrix} a & b & \alpha \\ c & d & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

where  $a, b, c, d \in \mathbb{F}_{2^n} \subset k$ ,  $ad + bc = 1$ ,  $\alpha, \beta \in k$ . We have an exact sequence of groups

$$(1) \quad 0 \rightarrow N \rightarrow G \rightarrow \mathrm{SL}_2(\mathbb{F}_{2^n}) \rightarrow 1,$$

where  $N$  is the kernel of the natural restriction map. In our basis,  $N$  consists of the matrices

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

where the column  $(\alpha, \beta)^T$  varies in some finite subset  $\Lambda$  of  $k^2$ . Denote by  $\Lambda_1$  the projection of  $\Lambda$  to the first coordinate.

**Lemma 2.1.** *The sets  $\Lambda$  and  $\Lambda_1$  have natural structures of vector spaces over the Galois field  $\mathbb{F}_{2^n}$ . Moreover  $\Lambda = (\Lambda_1, \Lambda_1)^T$  and  $\dim_{\mathbb{F}_{2^n}} \Lambda = 2 \dim_{\mathbb{F}_{2^n}} \Lambda_1$ .*

*Proof.* Obviously,  $N$  is an abelian group, and thus  $\Lambda$  is a subgroup of  $k^2$ . It remains to show that  $\Lambda$  is preserved by multiplication by an element  $e \in \mathbb{F}_{2^n}$ . Note that, as always in extensions with abelian  $N$ , the quotient group  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  acts on  $N$  via conjugation. In our case, this action is nothing else but the left multiplication of a column  $(\alpha, \beta)^T$  by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_{2^n}).$$

So, we have

$$\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_{2^n}), \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \Lambda \Rightarrow \\ \begin{pmatrix} \alpha + e\beta \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ e\alpha + \beta \end{pmatrix} \in \Lambda \Rightarrow e \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \in \Lambda.$$

But

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_{2^n}) \Rightarrow e \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \Lambda.$$

Multiplying a column  $(\alpha, \beta)^T \in \Lambda$  by matrices from the subgroup  $\mathrm{SL}_2(\mathbb{F}_2) < \mathrm{SL}_2(\mathbb{F}_{2^n})$ , one readily checks that the set  $\Lambda$  also contains  $(\alpha, 0)^T$ ,  $(0, \beta)^T$ ,  $(0, \alpha)^T$ , and  $(\beta, 0)^T$ . The remaining statements follow directly from this fact.  $\square$

The following proposition describes a convenient set of generators of the group  $\mathrm{SL}_2(\mathbb{F}_{2^n})$ .

**Proposition 2.2.** *The group  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  is generated by the matrices*

$$R = \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix}, S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

where  $e$  is a generator of the multiplicative group  $\mathbb{F}_{2^n}^*$  of the field  $\mathbb{F}_{2^n}$ .

*Proof.* It is well known (see, e.g., [2, Chapter 1]) that  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  is generated by its subgroup of diagonal matrices, the subgroup of upper triangular unipotent matrices, and the element

$$STS = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we are given the elements  $R, S, T$ , we can get any matrix

$$\begin{pmatrix} 1 & e^r \\ 0 & 1 \end{pmatrix}$$

as  $R^{-r/2}SR^{r/2}$ , where

$$R^{r/2} = \begin{pmatrix} e^{-r/2} & 0 \\ 0 & e^{r/2} \end{pmatrix}$$

(recall that each element of  $\mathbb{F}_{2^n}$  has a unique square root in  $\mathbb{F}_{2^n}$ ).  $\square$

*Remark 2.3.* Note that the matrices  $S$  and  $T$  generate the group  $\mathrm{SL}_2(\mathbb{F}_2)$ .

In our next step we show that sequence (1) splits.

**Lemma 2.4.** *After a change of the basis vector  $e_3$ , if necessary, we can assume that the group  $G$  contains matrices*

$$\tilde{S} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tilde{T} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and one of the matrices

$$\tilde{R} = \begin{pmatrix} e^{-1} & 0 & 1 \\ 0 & e & e \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \tilde{R}' = \begin{pmatrix} e^{-1} & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* As was shown in [8, Lemma 4.4], the group  $G$  contains transvections  $\tilde{S}$  and  $\tilde{T}$  that restrict to the elements  $S$  and  $T$  of  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  respectively. Each of the transvections  $\tilde{S}$  and  $\tilde{T}$  fixes a plane, and these planes intersect along a line not contained in the invariant subspace  $W$ . If we take  $e_3$  to be any non-zero vector from this line, then, in the basis  $e_1, e_2, e_3$ ,  $\tilde{S}$  and  $\tilde{T}$  have the desired matrices.

Now consider any element

$$\begin{pmatrix} e^{-1} & 0 & \alpha \\ 0 & e & \beta \\ 0 & 0 & 1 \end{pmatrix} \in G$$

that restricts to  $R \in \mathrm{SL}_2(\mathbb{F}_{2^n})$ . Using the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \tilde{S}\tilde{T}\tilde{S},$$

we get one more matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 & \alpha \\ 0 & e & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e & 0 & \beta \\ 0 & e^{-1} & \alpha \\ 0 & 0 & 1 \end{pmatrix} \in G,$$

thus

$$\begin{pmatrix} e^{-1} & 0 & \alpha \\ 0 & e & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e & 0 & \beta \\ 0 & e^{-1} & \alpha \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & e^{-1}\beta + \alpha \\ 0 & 1 & e\alpha + \beta \\ 0 & 0 & 1 \end{pmatrix} \in N.$$

Further,

$$\begin{pmatrix} 1 & 0 & e^{-1}\beta + \alpha \\ 0 & 1 & e\alpha + \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-1} & 0 & \alpha \\ 0 & e & \beta \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} e^{-2} & 0 & e^{-1}(\alpha + \beta) \\ 0 & e^2 & e(\alpha + \beta) \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

The  $2^{n-1}$ -th power of the last matrix equals

$$\begin{pmatrix} e^{-1} & 0 & (e^{1-2^n} + e^{3-2^n} + \cdots + e^{-1})(\alpha + \beta) \\ 0 & e & (e^{2^n-1} + e^{2^n-3} + \cdots + e)(\alpha + \beta) \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-1} & 0 & (e+1)^{-1}(\alpha + \beta) \\ 0 & e & e(e+1)^{-1}(\alpha + \beta) \\ 0 & 0 & 1 \end{pmatrix}.$$

If  $\alpha + \beta = 0$ , then we have found the matrix  $\tilde{R}' \in G$ . If  $\alpha + \beta \neq 0$ , then, rescaling the basis vector  $e_3$ , we come to the matrix  $\tilde{R} \in G$ .  $\square$

**Lemma 2.5.** *Let  $f: \mathbb{F}_{2^n}^2 \rightarrow \mathbb{F}_{2^n}$  be a function defined by the formula*

$$f(x, y) = 1 + x + y + x^{2^{n-1}}y^{2^{n-1}}.$$

*Then, for all  $a, b, c, d, p, q \in \mathbb{F}_{2^n}$  such that  $ad + bc = 1$ , the following identity holds:*

$$pf(a, b) + qf(c, d) + f(p, q) = f(pa + qc, pb + qd).$$

*Proof.* The lemma is proven by a straightforward substitution, bearing in mind that for any  $x \in \mathbb{F}_{2^n}$  one has  $x^{2^n} = x$ .  $\square$

**Corollary 2.6.** *For all  $\gamma \in k$  the set of matrices*

$$H_\gamma = \left\{ \begin{pmatrix} a & b & \gamma f(a, b) \\ c & d & \gamma f(c, d) \\ 0 & 0 & 1 \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_{2^n}) \right\}$$

*is a subgroup of  $\mathrm{GL}(V)$  isomorphic to  $\mathrm{SL}_2(\mathbb{F}_{2^n})$ .*

*Remark 2.7.* For any  $\gamma \in k$ , the map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} \gamma f(a, b) \\ \gamma f(c, d) \end{pmatrix}$$

is a skew homomorphism from the group  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  to the additive group  $k^2$ , generating the cohomology group  $H^1(\mathrm{SL}_2(\mathbb{F}_{2^n}), k^2)$ , where  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  acts on the space  $k^2$  of column vectors by left multiplication, see [4].

**Proposition 2.8.** *The group  $G$  contains one of the groups  $H_0$  or  $H_1$  defined in Corollary 2.6. It follows, in particular, that  $G$  is a semidirect product of the subgroups  $N$  and  $H_0$  ( $H_1$ ), that is, sequence (1) splits.*

*Proof.* Indeed, it can be directly checked that  $\tilde{R}, \tilde{S}, \tilde{T} \in H_1$ , whereas  $\tilde{R}', \tilde{S}, \tilde{T} \in H_0$ .  $\square$

*Remark 2.9.* It is known that the second cohomology group  $H^2(\mathrm{SL}_2(\mathbb{F}_{2^n}))$  with coefficients in the natural module is non-zero for  $n > 2$  ([3, Proposition 4.4]), i.e., there exist non-split extensions of  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  by  $\mathbb{F}_{2^n}^2$ . Our results mean that those non-split extensions do not have representations of the type that we study in this section.

*Remark 2.10.* Note that the groups  $H_0$  and  $H_1$  are defined over the field  $\mathbb{F}_{2^n}$ , i.e., the entries of all the matrices of  $H_0$  and  $H_1$  belong to  $\mathbb{F}_{2^n}$ .

### 3. PROOF OF THEOREM 1.3: INVARIANTS

In this section we compute the invariants of the action of the group  $G$  on the space  $V \simeq k^3$ . We do this in two steps: first, we compute the invariants of the kernel  $N$  and show that  $V/N$  is again isomorphic to  $k^3$ ; then, we compute the action of the quotient group  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  ( $\simeq H_0$  or  $H_1$ , see Proposition 2.8) on the invariants of  $N$  and show that also

$$V/G \simeq \frac{V/N}{H_0(H_1)} \simeq k^3.$$

We shall use the following criterion of Kemper.

**Proposition 3.1** ([6, Proposition 16]). *Let  $V$  be a vector space of dimension  $n$  and  $G < \mathrm{GL}(V)$  a finite group. Then  $S(V^*)^G$  is polynomial if and only if there exist homogeneous invariants  $f_1, \dots, f_n \in S(V^*)^G$  of degrees  $d_1, \dots, d_n$  such that  $\prod_{i=1}^n d_i = |G|$  and the Jacobian determinant  $J = \det((\partial f_i / \partial x_j)_{i,j})$  is non-zero. If such  $f_1, \dots, f_n$  exist, then they generate freely the ring  $S(V^*)^G$ .*

Recall that  $N$  acts on  $V$  by matrices

$$\begin{pmatrix} 1 & 0 & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix},$$

where the column  $(\alpha, \beta)^T$  runs over a finite dimensional  $\mathbb{F}_{2^n}$ -vector space  $\Lambda \subset k^2$ . Let  $x, y, z$  be a basis of  $V^*$  dual to the basis  $e_1, e_2, e_3$  of  $V$  chosen in Section 2. Obviously, the polynomials

$$\begin{aligned} f_x &= \prod_{\alpha \in \Lambda_1} (x + \alpha z), \\ f_y &= \prod_{\alpha \in \Lambda_1} (y + \alpha z), \\ f_z &= z \end{aligned}$$

are invariant under the action of  $N$ .

**Lemma 3.2.** *The polynomial  $f_x$  ( $f_y$ ) can involve  $x$  ( $y$ ) only in degrees  $2^{mn}$ , where  $0 \leq m \leq d = \dim_{\mathbb{F}_{2^n}} \Lambda_1$ .*

*Proof.* Let  $q = 2^n$  and

$$f'_x = \prod_{\alpha \in \Lambda_1} (x + \alpha).$$

By the definition of the Dickson invariants  $c_m \in k$  (see, e.g., [1, Section 8.1]), we have

$$f'_x = x^{q^d} + \sum_{m=0}^{d-1} c_m x^{q^m}.$$

To conclude the proof, it remains to note that  $f_x$  is obtained from  $f'_x$  by “homogenization” with the help of  $z$ : a monomial  $x^k$  with  $k \leq q^d$  is replaced by  $x^k z^{q^d - k}$ .  $\square$

**Proposition 3.3.** *The ring of invariants  $S(V^*)^N$  is a polynomial ring generated by  $f_x, f_y, f_z$ .*

*Proof.* We have  $|N| = |\Lambda| = 2^{2dn} = \deg f_x \cdot \deg f_y \cdot \deg f_z$ . Thus, by Proposition 3.1, we need to check only that the Jacobian is non-zero. But, using Lemma 3.2, we get

$$J(f_x, f_y, f_z) = \begin{vmatrix} \left( \prod_{\substack{\alpha \in \Lambda_1 \\ \alpha \neq 0}} \alpha \right) \cdot z^{2^{dn}-1} & 0 & \frac{\partial f_x}{\partial z} \\ 0 & \left( \prod_{\substack{\alpha \in \Lambda_1 \\ \alpha \neq 0}} \alpha \right) \cdot z^{2^{dn}-1} & \frac{\partial f_y}{\partial z} \\ 0 & 0 & 1 \end{vmatrix} = \left( \left( \prod_{\substack{\alpha \in \Lambda_1 \\ \alpha \neq 0}} \alpha \right) \cdot z^{2^{dn}-1} \right)^2 \neq 0.$$

□

Next we have to determine the action of the groups  $H_0$  and  $H_1$  on  $f_x$ ,  $f_y$ , and  $f_z$ . Let us begin with  $H_0$ . The generators of this group leave invariant the variable  $z$  and are defined over the field  $\mathbb{F}_{2^n}$  (see Remark 2.10). From this and from Lemma 3.2 it follows that the action of  $H_0$  on  $f_x$ ,  $f_y$ ,  $f_z$  is linear, that is, if  $h \in H_0$ , then it acts on the tuple  $(f_x, f_y, f_z)$  by right matrix multiplication:

$$(f_x, f_y, f_z) \mapsto (f_x, f_y, f_z) \cdot h.$$

Therefore, in this case we can simply ignore the kernel  $N$ . Furthermore, since (the representation of) the group  $H_0$  is decomposable, the polynomiality of its ring of invariants has been already established by Kemper and Malle [7, Section 8].

Now, consider the indecomposable group  $H_1$ . For the sake of clearness and simplicity, let us start with the case when there is no kernel, i.e.,  $N = \{0\}$  and  $H_1 = G$ . We shall need the invariants of the action of  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  on its natural module. Let  $W = k^2$  be a 2-dimensional space of column vectors over a field  $k$  containing  $\mathbb{F}_{2^n}$ , and let the group  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  act on  $W$  by left matrix multiplication. Denote by  $W^*$  the dual space. The Dickson invariants (see, e.g., [1, Proposition 8.1.3]) are

$$c_0 = \prod_{\substack{l \in W^* \\ l \neq 0}} l,$$

and

$$c_1 = \sum_{\substack{U \subseteq W \\ \dim U = 1}} \prod_{\substack{l \in W^* \\ l|_U \neq 0}} l$$

(for  $c_1$  the sum is taken over all 1-dimensional subspaces of  $W$ , and the product over all linear forms that restrict to a non-zero form on  $U$ ). It is not hard to see that there exists a root of degree  $2^n - 1$  of the polynomial  $c_0$ , that is,  $\exists u \in S(W^*) : u^{2^n-1} = c_0$ , and that  $u$  and  $c_1$  are  $\mathrm{SL}_2(\mathbb{F}_{2^n})$ -invariant.

**Theorem 3.4** ([1, Theorem 8.2.1]). *The ring of invariants of  $\mathrm{SL}_2(\mathbb{F}_{2^n})$  on  $W$  is polynomial and generated by  $u$  and  $c_1$ .*

Let us come back to our group  $G = H_1$  and space  $V$ . Since we have a  $G$ -invariant subspace  $W$ , the restriction to  $W$  of each invariant of  $G$  is an  $\mathrm{SL}_2(\mathbb{F}_{2^n})$ -invariant. Thus we have a homomorphism  $S(V^*)^G \rightarrow S(W^*)^{\mathrm{SL}_2(\mathbb{F}_{2^n})}$  of invariant rings. In a general modular case, there is no reason for such a homomorphism to be surjective. However, we shall see that we do have a surjection in our case and this will be a crucial step in computing the invariants of  $G$ .

**Lemma 3.5.** *Let  $G$ ,  $V$ , and  $W$  be as defined above. Then the restriction homomorphism  $S(V^*)^G \rightarrow S(W^*)^{\mathrm{SL}_2(\mathbb{F}_{2^n})}$  is surjective.*

*Proof.* It is sufficient to lift to the ring  $S(V^*)^G$  the invariants  $u, c_1 \in S(W^*)^{\mathrm{SL}_2(\mathbb{F}_{2^n})}$ . We shall work in the explicit coordinates  $x, y, z$  defined after Proposition 3.1, so that any linear form  $l \in V^*$  can be written as  $l = ax + by + cz$ ,  $a, b, c \in k$ . Together with the function  $f$  (see Lemma 2.5), consider also a function  $g: \mathbb{F}_{2^n}^2 \rightarrow \mathbb{F}_{2^n}$ :

$$g(x, y) = f(x, y) + 1 = x + y + x^{2^{n-1}}y^{2^{n-1}}.$$

It follows from Lemma 2.5 that  $g$  has the following property: for all  $a, b, c, d, p, q \in \mathbb{F}_{2^n}$ , if  $ad + bc = 1$ , then

$$(2) \quad pf(a, b) + qf(c, d) + g(p, q) = g(pa + qc, pb + qd).$$

Note also that  $g$  is a homogeneous function of degree 1 on  $\mathbb{F}_{2^n}^2$ , i.e.,

$$(3) \quad \forall a, b, t \in \mathbb{F}_{2^n} \quad g(ta, tb) = tg(a, b).$$

Now, let us lift each linear form  $l = ax + by \in W^*$  to  $V^*$  by the formula  $\tilde{l} = ax + by + g(a, b)z$  and define

$$\begin{aligned} \tilde{c}_0 &= \prod_{\substack{l \in W^* \\ l \neq 0}} \tilde{l}, \\ \tilde{c}_1 &= \sum_{\substack{U \subseteq W \\ \dim U = 1}} \prod_{\substack{l \in W^* \\ l|_U \neq 0}} \tilde{l}. \end{aligned}$$

Property (2) implies that both  $\tilde{c}_0$  and  $\tilde{c}_1$  are  $G$ -invariant. Obviously,  $\tilde{c}_0|_W = c_0$ ,  $\tilde{c}_1|_W = c_1$ . But, using property (3), one readily shows that  $\tilde{c}_0$  admits a root of degree  $2^n - 1$ , i.e., there exists  $\tilde{u} \in S(V^*)$  such that  $\tilde{u}^{2^n - 1} = \tilde{c}_0$ . Moreover, this  $\tilde{u}$  is  $G$ -invariant and restricts to  $u \in S(W^*)^{\mathrm{SL}_2(\mathbb{F}_{2^n})}$ .  $\square$

**Proposition 3.6.** *The ring of invariants  $S(V^*)^G$  (for  $G = H_1$ ) is polynomial and generated by (algebraically independent) invariants  $\tilde{u}$ ,  $\tilde{c}_1$ ,  $z$ , where  $\tilde{u}$  and  $\tilde{c}_1$  are defined in the proof of Lemma 3.5.*

*Proof.* Let  $\tilde{c} \in S(V^*)^G$  be an arbitrary homogeneous invariant. Let  $c = \tilde{c}|_W$ . Write  $c$  as a polynomial of  $u$  and  $c_1$ :

$$c = h(u, c_1).$$

The  $G$ -invariant  $\tilde{c} - h(\tilde{u}, \tilde{c}_1)$  vanishes on  $W$ , thus it is divisible by  $z$ . But since  $z$  is also a  $G$ -invariant, so is the polynomial

$$\tilde{c}' = (\tilde{c} - h(\tilde{u}, \tilde{c}_1))/z.$$

The degree of  $\tilde{c}'$  is strictly less than that of  $\tilde{c}$ , so, proceeding by induction, we express  $\tilde{c}$  through  $\tilde{u}$ ,  $\tilde{c}_1$ , and  $z$ .  $\square$

Now we return to the general case of a non-zero kernel  $N$ . A direct calculation with a use of Lemma 3.2 shows that the two generators  $\tilde{S}$ ,  $\tilde{T}$  (see Lemma 2.4) of the group  $H_1$  act on the basis invariants  $f_x, f_y, f_z$  of  $N$  by the formulae

$$\begin{aligned} f_x \cdot \tilde{S} &= f_x + f_y, & f_y \cdot \tilde{S} &= f_y, & f_z \cdot \tilde{S} &= f_z, \\ f_x \cdot \tilde{T} &= f_x, & f_y \cdot \tilde{T} &= f_x + f_y, & f_z \cdot \tilde{T} &= f_z, \end{aligned}$$

i.e., their action is linear. It follows from Lemma 3.2 that the third generator  $\tilde{R}$  acts by the formulae

$$f_x \cdot \tilde{R} = e^{-1}f_x + \alpha z^{2^{dn}}, \quad f_y \cdot \tilde{R} = ef_x + e\alpha z^{2^{dn}}, \quad f_z \cdot \tilde{R} = f_z,$$

where  $\alpha \in k$ . It can happen that  $\alpha = 0$ , so that the action of  $H_1$  on  $V/N$  is linear (in coordinates  $f_x, f_y, f_z$ ) and decomposable. But then again by the results of



Kemper and Malle the ring of invariants  $S((V/N)^*)^{H_1} = S(V^*)^G$  is polynomial. In general, the coefficient  $\alpha$  does not vanish and the action of  $\tilde{R}$  becomes non-linear. Still, it is possible to follow the argument of Lemma 3.5 and Proposition 3.6.

Note that the equation  $f_z = z = 0$  defines an invariant subspace  $W/N$  of the quotient  $V/N$  (which we consider as a vector space isomorphic to  $k^3$ , the isomorphism being defined by the functions  $f_x, f_y, f_z$ ). The action of  $H_1$  on  $W/N$  is the natural action of  $\mathrm{SL}_2(\mathbb{F}_{2^n})$ . So, let  $u$  and  $c_1$  be the basis invariants of  $\mathrm{SL}_2(\mathbb{F}_{2^n})$ , but now considered as functions of  $f_x, f_y, f_z$ . Repeating the proof of Lemma 3.5 with  $f_x$  in place of  $x$ ,  $f_y$  in place of  $y$ , and  $\alpha z^{2^{d_n}}$  in place of  $z$ , we find some liftings  $\bar{u}$  and  $\bar{c}_1$  of  $u$  and  $c_1$  to the ring of invariants  $S(V^*)^G$ .

The following proposition finishes the proof of Theorem 1.3.

**Proposition 3.7.** *The ring of invariants  $S(V^*)^G = S((V/N)^*)^{H_1}$  is polynomial and generated by (algebraically independent) invariants  $\bar{u}, \bar{c}_1, z$ .*

*Proof.* This proposition is proven by argument similar to the proof of Proposition 3.6.  $\square$

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